

# LOW-ORDER HAMILTONIAN OPERATORS HAVING MOMENTUM

JIŘINA VODOVÁ

**ABSTRACT.** We describe all fifth-order Hamiltonian operators in one dependent and one independent variable that possess the momentum, i.e., for which there exists a Hamiltonian associated with translation in the independent variable. Similar results for first- and third-order Hamiltonian operators were obtained earlier by Mokhov.

## 1. INTRODUCTION

The Hamiltonian evolution equations are well known to play an important role in modern mathematical physics [3, 5, 10, 12]. Indeed, a Hamiltonian operator maps the variational derivatives of the conserved quantities into symmetries; this is of particular significance in the theory of integrable systems which often turn out to be *bi*-Hamiltonian, see e.g. [4, 5, 8, 9, 13, 14] and references therein.

In this paper we employ the so-called special contact transformations, introduced in [11], to classify fifth-order Hamiltonian operators admitting momentum, see below for details. Special contact transformations preserve existence of momentum, and for this reason we study the existence of momentum for just the representatives of the associated equivalence classes.

Existence of momentum is useful for averaging the corresponding Hamiltonian systems, see e.g. [7]. Operators having momentum could be employed e.g. for the generation of hierarchies of local symmetries (i.e., higher commuting flows) in the following fashion.

Suppose we are given a Hamiltonian operator in one dependent variable  $u$  and one independent variable  $x$ , say  $\mathfrak{D}$ , possessing momentum, i.e., there exists a functional  $\mathcal{P} = \int h dx$  such that  $u_x = \mathfrak{D}\delta_u \mathcal{P}$ . Further assume that there exists another translation-invariant Hamiltonian operator  $\mathfrak{E}$  which is compatible with  $\mathfrak{D}$  and the operator  $\mathfrak{R} := \mathfrak{E} \circ \mathfrak{D}^{-1}$  is weakly nonlocal hereditary operator. Then  $\mathfrak{R}$  is a recursion operator for the equation  $u_{t_0} = u_x$ , and under a further minor technical assumption of normality of  $\mathfrak{R}$  in the sense of [18], by Theorem 1 from [18] the quantities  $\mathfrak{R}^i(u_x)$  are local and the associated flows commute for all  $i = 1, 2, 3, \dots$ , i.e., we have an infinite hierarchy of local commuting flows  $u_{t_j} = \mathfrak{R}^j(u_x)$ ,  $j = 0, 1, 2, \dots$ .

## 2. PRELIMINARIES

In what follows we are going to deal with Hamiltonian operators and associated Hamiltonian evolution equations involving a single spatial variable  $x$  and a single dependent variable  $u$ . A Hamiltonian evolution equation takes the form

$$u_t = \mathfrak{D}\delta_u \mathcal{T}[u],$$

---

2010 *Mathematics Subject Classification.* 37K05, 37K10.

*Key words and phrases.* Hamiltonian operators, evolution equations, averaging.

where  $\mathfrak{D}$  is a Hamiltonian operator,  $\mathcal{T} = \int T[u]dx$  is a functional (often referred to as the Hamiltonian), and  $\delta_u$  denotes the variational derivative with respect to  $u$ . Recall (see e.g. [13] for details) that the operator  $\mathfrak{D}$  defines a Poisson bracket:

$$\{\mathcal{R}, \mathcal{S}\} = \int \delta_u \mathcal{R} \mathfrak{D} \delta_u \mathcal{S} dx,$$

and thus satisfies certain requirements that coincide with the condition of *skew symmetry*:

$$\{\mathcal{R}, \mathcal{S}\} = -\{\mathcal{S}, \mathcal{R}\},$$

and the *Jacobi identity*:

$$\{\{\mathcal{R}, \mathcal{S}\}, \mathcal{T}\} + \{\{\mathcal{S}, \mathcal{T}\}, \mathcal{R}\} + \{\{\mathcal{T}, \mathcal{R}\}, \mathcal{S}\} = 0,$$

for all functionals  $\mathcal{R}, \mathcal{S}$ , and  $\mathcal{T}$ .

It can be shown that the skew-symmetry condition is equivalent to the skew-adjointness of the operator  $\mathfrak{D}$ . Let  $D_x$  denote the total derivative with respect to the spatial variable  $x$  and  $u_i \equiv D_x^i(u)$ . Recall ([13]) that a *differential function* by definition depends on  $x$ ,  $u$ , and finitely many derivatives of  $u$  with respect to the space variable  $x$ .

In [5], for any operator  $\mathfrak{D} = \sum_{k=0}^N p_k D_x^k$  and for any differential function  $f$  the author defines another differential operator  $D_{\mathfrak{D}}f$  by the formula

$$(D_{\mathfrak{D}}f)h = (\text{pr } v_h(\mathfrak{D}))(f),$$

where  $\text{pr } v_h$  is the prolongation of a vector field  $v_h$  with the characteristic  $h$ , i.e.,

$$\text{pr } v_h = h \frac{\partial}{\partial u} + \sum_{i=1} D^i(h) \frac{\partial}{\partial u_i}.$$

We have

$$D_{\mathfrak{D}}f = \sum_{k,m} \frac{\partial p_k}{\partial u_m} D^k(f) D^m,$$

and it is proved that the Jacobi identity for the skew-adjoint operator  $\mathfrak{D}$  is equivalent to the condition

$$(D_{\mathfrak{D}}h_1)\mathfrak{D}h_2 - (D_{\mathfrak{D}}h_2)\mathfrak{D}h_1 + \mathfrak{D}(D_{\mathfrak{D}}h_1)^*h_2 = 0, \quad (1)$$

which must hold for arbitrary smooth differential functions  $h_1$  and  $h_2$ .

Recall (see e.g. [3]) that the *differential order* of a differential function  $f$ , denoted by  $\text{ord}(f)$  is the maximal  $m \in \mathbb{Z}_+$  such that  $\frac{\partial f}{\partial u_m} \neq 0$  if  $f$  is not a quasiconstant ( $f$  is *quasiconstant* if it depends only on the spatial variable  $x$ ), and is  $-\infty$  if  $f$  is quasiconstant. Following [5] define the *level*  $m$  of the Hamiltonian operator  $\mathfrak{D} = \sum_{k=0}^N p_k D^k$  of order  $N$  to be  $m = \max_j \{j + \text{ord}(p_j)\}$ . The possible values of the level of a nonquasiconstant-coefficient Hamiltonian operator are treated in [3]. In this paper we are specifically interested in fifth-order nonquasiconstant-coefficient Hamiltonian operators, whose only possible level values  $m$  are  $m = 5, 6$  or  $7$  [5].

Following [11] we say that a Hamiltonian operator  $\mathfrak{D}$  *has a momentum* if there exists a functional  $\mathcal{T}$  such that

$$\mathfrak{D}\delta_u \mathcal{T} = u_1.$$

Differential substitutions are one of the most important tools using which we can distinguish Hamiltonian operators having momentum from those that have none. The next lemma shows how the Hamiltonian operators behave under differential substitutions:

**Lemma 1** ([11]). *Let  $\mathfrak{D}_1$  be a Hamiltonian operator in the variables  $x, u$ . Under the transformation*

$$x = \varphi(y, v, v_1, \dots, v_m), \quad u = \psi(y, v, v_1, \dots, v_n), \quad (2)$$

*where  $v_j = D_y^j(v)$ , and  $D_y$  is the total derivative with respect to  $y$ , the operator  $\mathfrak{D}_1$  goes into the Hamiltonian operator  $\mathfrak{D}_2$  defined by the formula*

$$\overline{\mathfrak{D}}_1 = (D_y(\varphi))^{-1} K^* \circ \mathfrak{D}_2 \circ K, \quad (3)$$

*where*

$$K = \sum_{i=0}^{\max(m,n)} (-1)^i D_y^i \circ \left( \frac{\partial \psi}{\partial v_i} D_y(\varphi) - \frac{\partial \varphi}{\partial v_i} D_y(\psi) \right),$$

*$K^*$  is the formal adjoint of  $K$ , and  $\overline{\mathfrak{D}}_1$  is obtained from  $\mathfrak{D}_1$  upon using (2) and setting  $D_x = (D_y(\varphi))^{-1} D_y$ .*

**Remark 1.** Note that in general the operator  $\mathfrak{D}_2$  may contain nonlocal terms unless (2) is a contact transformation, cf. e.g. [1, 3, 11].

General contact transformations do not preserve the property of having momentum. However, in [11] Mokhov introduced a pseudogroup of *special contact transformations*

$$x = \varphi(y, v, v_y) = y + w(v, v_y), \quad u = \psi(v, v_y),$$

$$\frac{\partial \varphi}{\partial v_y} D_y(\psi) = \frac{\partial \psi}{\partial v_y} D_y(\varphi), \quad \rho = \frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial v} D_y(\psi) / D_y(\varphi) \neq 0$$

which preserve existence of momentum.

It is readily checked that the coefficients of Hamiltonian operators having momentum may not explicitly depend on the spatial variable  $x$ , that is, the Hamiltonian operators having momentum are to be found among the translation-invariant ones. This happens because in order to possess momentum the Lie derivative of such an operator along the vector field with the characteristic  $u_1$  must vanish.

### 3. FIRST- AND THIRD-ORDER HAMILTONIAN OPERATORS HAVING MOMENTUM

Mokhov ([11]) has shown that for the first-order Hamiltonian operators the condition of translation invariance is not only necessary but also sufficient for the existence of momentum:

**Proposition 1** ([11]). *A first-order Hamiltonian operator has a momentum if and only if it is translation-invariant.*

Classification of the third-order translation-invariant operators under a special contact transformation was also obtained by Mokhov. He employed it to find out the question whether a given third-order translation-invariant Hamiltonian operator has a momentum or not:

**Proposition 2** ([11]). *An arbitrary translation-invariant Hamiltonian operator of the third order can be reduced by a special contact transformations to one of the operators (4)-(6).*

(1) *An operator*

$$\mathfrak{D} = \pm \frac{1}{u_x} [D_x^3 + 2SD_x + D_x S] \circ \frac{1}{u_x} + 2fD_x + D_x f, \quad (4)$$

where  $S = \frac{u_3}{u_1} - \frac{3}{2} \frac{(u_2)^2}{(u_1)^2}$  and  $f$  is an arbitrary function of  $u$  only, has a momentum. The corresponding functional is of the form  $\int p(u)dx$ , where  $p(u)$  is the solution of the equation

$$\pm \frac{\partial^4 p}{\partial u^4} + 2f(u) \frac{\partial^2 p}{\partial u^2} + \frac{\partial p}{\partial u} \frac{\partial f}{\partial u} - 1 = 0.$$

(2) *An operator*

$$\mathfrak{D} = \pm [D_x^3 + 2AuD_x + Au_x], A = \text{const} > 0 \quad (5)$$

has a momentum.

(3) *An operator*

$$\mathfrak{D} = \pm [D_x^3 + AD_x], A = \text{const}. \quad (6)$$

does not have momentum.

#### 4. FIFTH-ORDER HAMILTONIAN OPERATORS HAVING MOMENTUM

As proved above, one should look for Hamiltonian operators having momentum among the translation-invariant ones. We will classify the fifth-order translation-invariant operators according to their leading coefficients up to special contact transformations which preserve the property of having momentum:

**Proposition 3.** *Any fifth-order Hamiltonian operator can be reduced by a special contact transformation to an operator with leading coefficient equal to either  $\pm 1$  or  $\pm \frac{1}{u_1^4}$ .*

*Proof.* The proof partially uses the line of reasoning analogous to the one used by Mokhov in his classification of third-order operators. The leading coefficient of a fifth-order translation-invariant Hamiltonian operator has the general form (see [2])

$$\pm \frac{1}{(\alpha v_2 + \beta)^6}, \quad \alpha = \alpha(v, v_1), \quad \beta = \beta(v, v_1).$$

If  $\alpha \neq 0$ , then we can find a special contact transformation to get rid of the dependence of the leading coefficient of the operator on  $v_2$  in the following way: take a function  $\tilde{w}(v, v_1)$  such that  $\frac{\partial \tilde{w}}{\partial v} v_1 + 1 = \frac{\beta}{\alpha} \frac{\partial \tilde{w}}{\partial v_1}$ ,  $\frac{\partial \tilde{w}}{\partial v_1} \neq 0$ , and a function  $\tilde{\psi}(v, v_1) \neq 0$  such that  $(1 + D_y(\tilde{w})) \frac{\partial \tilde{\psi}}{\partial v_1} = \frac{\partial \tilde{w}}{\partial v_1} D_y(\tilde{\psi})$  and  $\tilde{\rho} := \frac{\partial \tilde{\psi}}{\partial v} - \frac{\partial \tilde{w}}{\partial v} \frac{D_y(\tilde{\psi})}{(1 + D_y(\tilde{w}))} \neq 0$ . The functions  $\tilde{w}$  and  $\tilde{\psi}$  define a special contact transformation:

$$x = y + \tilde{w}(v, v_1), \quad u = \tilde{\psi}(v, v_1), \quad (7)$$

where  $x$  is a new independent variable and  $u$  is a new dependent variable. The inverse of (7) is also a contact transformation:

$$y = x + w(u, u_1) = \varphi(x, u, u_1), \quad v = \psi(u, u_1), \quad (8)$$

and it can be verified that the leading coefficient of the operator transformed by (8) does not depend on  $u_2$ .

Now suppose that  $\alpha \equiv 0$ . Then [2] the leading coefficient of our operator is of the form  $\pm \frac{1}{(\tilde{\alpha}v_1 + \tilde{\beta})^4}$ ,  $\tilde{\alpha} = \tilde{\alpha}(v)$ ,  $\tilde{\beta} = \tilde{\beta}(v)$ . If  $\tilde{\beta} \equiv 0$ , there is no possibility to get rid of the dependence of the leading coefficient of the operator on  $v_1$  using only special contact transformations. Using the transformation

$$y = x + w(u), \quad v = \psi(u)$$

such that  $\frac{\partial \psi}{\partial u} = \sqrt[3]{1/\tilde{\alpha}^2}$  makes the leading coefficient equal to  $\pm \frac{1}{u_1^4}$ . If  $\tilde{\beta} \neq 0$ , a special contact transformation

$$y = x + w(u), \quad v = \psi(u)$$

which is an inverse of the transformation

$$x = y + \tilde{w}(v), \quad u = \tilde{\psi}(v) \neq \text{const.}, \quad \frac{\partial \tilde{w}}{\partial v} = \frac{\tilde{\alpha}}{\tilde{\beta}},$$

turns our operator into an operator with a leading coefficient that does not depend on  $u_1$ .

If the differential order of the leading coefficient is equal to zero (i.e. the leading coefficient of our translation-independent depends only on  $v$ , and is therefore of the form  $\frac{1}{\alpha(v)}$ ), the special contact transformation

$$y = x, \quad v = \psi(u), \quad \left( \frac{\partial \psi}{\partial u} \right)^2 = \pm \frac{1}{\alpha(v(u))}$$

makes the leading coefficient of our transformed operator equal to  $\pm 1$ . □

In what follows a fifth-order Hamiltonian operator is supposed to be written in the form

$$\mathfrak{D} = aD_x^5 + D_x^5 \circ a + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

which ensures skew-symmetry of the operator and hence skew-symmetry of the associated Poisson bracket.

Next, we are going to show that no fifth-order Hamiltonian operator with the leading coefficient  $\pm 1$  has momentum and find out what the functional  $\mathcal{P}$  providing the existence of momentum for the operators with the leading coefficient  $\pm 1/u_1^4$  looks like. Observe that the operators with the leading coefficient  $-1$  can be transformed by the special contact transformation  $x = y$ ,  $u = iv$  to operators with the leading coefficient  $1$  so it is sufficient to show that no fifth-order Hamiltonian operator with the leading coefficient  $1$  has momentum. The same reasoning could be applied to operators with the leading coefficient  $-1/u_1^4$ , but as we are interested in finding the explicit form of the desired functional  $\mathcal{P}$  for operators with the leading coefficients  $1/u_1^4$  and  $-1/u_1^4$ , we discuss each of these cases separately.

Thus, we first try to find general forms of fifth-order translation-invariant Hamiltonian operators with the leading coefficients  $1$ ,  $1/u_1^4$  and  $-1/u_1^4$ . The answer to this question in the case of Hamiltonian operator with the leading coefficient equal to  $1$  is given (even though in a more general setting than we actually need) in [2]:

**Lemma 2.** *A fifth-order Hamiltonian operator whose leading coefficient is 1 must be of the form*

$$\mathfrak{D} = D_x^5 + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where  $b$  and  $c$  are functions of  $x$  alone. Otherwise they are given by

$$\begin{aligned} b &= \frac{3}{2}(u + \alpha)^{-1}(u_{xx} + \alpha'') - \frac{7}{4}(u + \alpha)^{-2}(u_x + \alpha')^2 + \beta(u + \alpha) + \gamma, \\ c &= -\frac{z_4}{z} + \frac{\beta z_1^2}{2z} + \frac{wz_2}{2z} - \frac{wz_1^2}{4z^2} - \frac{w_1 z_1}{z} + \frac{9z_1 z_3}{2z^2} - \frac{129z_1^2 z_2}{8z^3} + \frac{273z_1^4}{32z^4} \\ &\quad + \frac{33z_2^2}{8z^2} - \frac{\beta z_2}{2} - \frac{3z\beta''}{2} - \frac{\beta' z_1}{2} - \frac{\beta^2 z^2}{2} + \frac{w^2}{2}, \end{aligned}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are functions of  $x$  only,  $w$  and  $z$  are given by

$$w = \beta z + \gamma, \quad z = u + \alpha,$$

and  $w_i = D_x^i(w)$ ,  $z_i = D_x^i(z)$ .

If  $\beta = 0$ , then any choice of  $\alpha$  and  $\gamma$  yields a Hamiltonian operator.

If  $\beta \neq 0$ , then

$$\gamma = -\frac{\rho}{\beta^2} - \frac{\beta''}{2\beta} + \frac{(\beta')^2}{4\beta^2},$$

where  $\rho$  is an arbitrary constant.

**Lemma 3.** *A fifth-order translation-invariant Hamiltonian operator whose leading coefficient is  $\pm 1/u_1^4$  must be of the form*

$$\mathfrak{D} = \pm \frac{1}{2u_1^4} D_x^5 \pm \frac{1}{2u_1^4} D_x^5 \circ 1 + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6} (\pm 10u_3u_1 \mp 55u_2^2 + 2\alpha u_1^4), \\ c &= \frac{1}{u_1^8} \left( 3u_1^6 u_2 \frac{\partial \alpha}{\partial u} + 2u_1^5 u_3 \alpha - 6u_1^4 u_2^2 \alpha + \beta u_1^8 \mp 3u_1^3 u_5 \pm 65u_1^2 u_2 u_4 \pm 50u_1^2 u_3^2 \right. \\ &\quad \left. \mp 615u_1 u_2^2 u_3 \pm 735u_2^4 \right), \end{aligned}$$

and  $\alpha$  and  $\beta$  are functions of  $u$  only.

**Remark 2.** It can be shown that there is no special contact transformation which preserves the leading coefficient and simultaneously eliminates one of the unknown functions  $\alpha$ ,  $\beta$ .

*Proof.* We will prove the assertion only for the case of the leading coefficient  $1/u_1^4$ . The proof for the case of the leading coefficient equal to  $-1/u_1^4$  can be obtained in a similar way. Put  $a = 1/(2u_1^4)$ . Then

the Jacobi identity implies the following relations (cf. [2]):

$$\frac{\partial c}{\partial u_6} = 0 \quad (9)$$

$$\frac{\partial b}{\partial u_4} = 0 \quad (10)$$

$$\frac{\partial c}{\partial u_5} = -\frac{3}{u_x^5} \quad (11)$$

$$\frac{\partial b}{\partial u_3} = \frac{5}{u_1^5} \quad (12)$$

$$\frac{\partial c}{\partial u_4} = \frac{1}{3u_1^6} \left( 85u_2 - 2\frac{\partial b}{\partial u_2}u_1^6 \right) \quad (13)$$

$$\frac{\partial c}{\partial u_3} = \frac{1}{3u_1^7} \left( -16\frac{\partial b}{\partial u_2}u_1^6u_2 - 225u_1u_3 - 9D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^7 + 410u_2^2 + 6bu_1^6 \right) \quad (14)$$

$$\frac{\partial b}{\partial u_1} = \frac{1}{3u_1^7} \left( 26\frac{\partial b}{\partial u_2}u_1^6u_2 + 340u_1u_3 + 7D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^7 - 550u_2^2 - 6bu_1^6 \right) \quad (15)$$

$$\begin{aligned} \frac{\partial c}{\partial u_2} = & \frac{1}{6u_1^8} \left( -3\frac{\partial b}{\partial u}u_1^8 + 140\frac{\partial b}{\partial u_2}u_1^6u_2^2 + 80bu_1^6u_2 + 11390u_1u_2u_3 - 96\frac{\partial b}{\partial u_2}u_1^7u_3 - 14260u_2^3 \right. \\ & \left. - 27D_x^2 \left( \frac{\partial b}{\partial u_2} \right) u_1^8 + 21D_x(b)u_1^7 - 1200u_1^2u_4 + 2\frac{\partial b}{\partial u_2}bu_1^{12} - 82D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^7u_2 \right) \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial c}{\partial u_1} = & \frac{1}{6u_1^9} \left( 18\frac{\partial b}{\partial u}u_1^8u_2 - 42D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^7u_2^2 + 416\frac{\partial b}{\partial u_2}u_1^6u_2^3 - 271D_x(b)u_1^7u_2 - 856bu_1^6u_2^2 \right. \\ & + 80bu_1^7u_3 + 14730u_1^2u_2u_4 - 214D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^8u_3 - 68\frac{\partial b}{\partial u_2}u_1^8u_4 - 136D_x^2 \left( \frac{\partial b}{\partial u_2} \right) u_1^8u_2 \\ & + 2D_x \left( \frac{\partial b}{\partial u_2} \right) bu_1^{13} - 4\frac{\partial b}{\partial u_2}D_x(b)u_1^{13} - 92450u_1u_2^2u_3 - 1080u_1^3u_5 - 21D_x^3 \left( \frac{\partial b}{\partial u_2} \right) u_1^9 \\ & \left. + 3D_x^2(b)u_1^8 + 7610u_1^2u_3^2 - 3D_x \left( \frac{\partial b}{\partial u} \right) u_1^9 - 404\frac{\partial b}{\partial u_2}u_1^7u_2u_3 - 4\frac{\partial b}{\partial u_2}bu_1^{12}u_2 + 87920u_2^4 \right) \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial c}{\partial u} = & \frac{1}{6u_1^{10}} \left( -21D_x^4 \left( \frac{\partial b}{\partial u_2} \right) u_1^{10} + 9D_x^2 \left( \frac{\partial b}{\partial u} \right) u_1^{10} + 28410u_1^3u_3u_4 + 15730u_1^3u_2u_5 - 66D_x^2(b)u_1^8u_2 \right. \\ & - 608D_x(b)u_1^7u_2^2 - 717D_x(b)u_1^8u_3 - 13280bu_1^6u_2^3 - 660bu_1^8u_4 - 205510u_1^2u_2u_3^2 - 139340u_1^2u_2^2u_4 \\ & + 838900u_1u_2^3u_3 - 80D_x(b)\frac{\partial b}{\partial u_2}u_1^{13}u_2 - 1416\frac{\partial b}{\partial u_2}u_1^7u_2^2u_3 - 2062D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^8u_2u_3 \\ & - 464\frac{\partial b}{\partial u_2}bu_1^{12}u_2^2 - 32\frac{\partial b}{\partial u_2}bu_1^{13}u_3 - 120D_x \left( \frac{\partial b}{\partial u_2} \right) bu_1^{13}u_2 - 744\frac{\partial b}{\partial u_2}u_1^8u_2u_4 - 673120u_2^5 \\ & + 440cu_1^8u_2 - 6D_x(b)bu_1^{13} - 232b^2u_1^{12}u_2 - 1092u_1^4u_6 - 9D_x^3(b)u_1^9 + 6D_x(c)u_1^9 \\ & - 252D_x^3 \left( \frac{\partial b}{\partial u_2} \right) u_1^9u_2 - 956D_x^2 \left( \frac{\partial b}{\partial u_2} \right) u_1^8u_2^2 - 408D_x^2 \left( \frac{\partial b}{\partial u_2} \right) u_1^9u_3 + 8c\frac{\partial b}{\partial u_2}u_1^{14} + 6\frac{\partial b}{\partial u}bu_1^{14} \\ & - 4b^2\frac{\partial b}{\partial u_2}u_1^{18} - 1304D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^7u_2^3 - 282D_x \left( \frac{\partial b}{\partial u_2} \right) u_1^9u_4 - 2200\frac{\partial b}{\partial u_2}u_1^6u_2^4 - 68\frac{\partial b}{\partial u_2}u_1^9u_5 \\ & + 66D_x \left( \frac{\partial b}{\partial u} \right) u_1^9u_2 + 192\frac{\partial b}{\partial u}u_1^8u_2^2 + 12\frac{\partial b}{\partial u}u_1^9u_3 - 12D_x^2 \left( \frac{\partial b}{\partial u_2} \right) bu_1^{14} - 12D_x \left( \frac{\partial b}{\partial u_2} \right) D_x(b)u_1^{14} \\ & \left. - 708\frac{\partial b}{\partial u_2}u_1^8u_3^2 + 3124bu_1^7u_2u_3 \right). \end{aligned} \quad (18)$$

We can equate mixed partial derivatives to obtain new relations. Equating  $\frac{\partial}{\partial u} \left( \frac{\partial c}{\partial u_4} \right)$  and  $\frac{\partial}{\partial u_4} \left( \frac{\partial c}{\partial u} \right)$ , evaluating the total derivatives and substituting for the partial derivatives of  $c$  with respect to  $u_6, u_5, u_4, u_3, u_2$  and for the partial derivative of  $b$  with respect to  $u_4$  and  $u_3$  from the relations (9)-(14) and (16) gives

$$\begin{aligned}
0 = & 1197u_1^8 \frac{\partial^2 b}{\partial u_2 \partial u} + 252u_1^9 \frac{\partial^3 b}{\partial u_2 \partial u_1 \partial u} + 2034bu_1^6 + 378u_1^{10} \frac{\partial^4 b}{\partial u_2^2 \partial u^2} - 71865u_3u_1 + 27u_1^7 \frac{\partial b}{\partial u_1} \\
& + 16u_1^{12} \left( \frac{\partial b}{\partial u_2} \right)^2 + 756u_1^8 u_2 u_3 \frac{\partial^4 b}{\partial u_2^3 \partial u_1} + 392590u_2^2 + 2268u_1^7 u_2 u_3 \frac{\partial^3 b}{\partial u_2^3} + 378u_1^8 u_2^2 \frac{\partial^4 b}{\partial u_2^2 \partial u_1^2} \\
& + 36bu_1^{12} \frac{\partial^2 b}{\partial u_2^2} + 2868u_1^6 u_2^2 \frac{\partial^2 b}{\partial u_2^2} + 756u_1^9 u_3 \frac{\partial^4 b}{\partial u_2^3 \partial u} + 3926u_1^6 u_2 \frac{\partial b}{\partial u_2} + 756u_1^9 u_2 \frac{\partial^4 b}{\partial u_2^2 \partial u_1 \partial u} \\
& + 378u_1^8 u_3^2 \frac{\partial^4 b}{\partial u_2^4} + 630u_1^8 u_3 \frac{\partial^3 b}{\partial u_2^2 \partial u_1} + 252u_1^8 u_2 \frac{\partial^3 b}{\partial u_2 \partial u_1^2} + 1929u_1^7 u_2 \frac{\partial^2 b}{\partial u_2 \partial u_1} + 2268u_1^7 u_2^2 \frac{\partial^3 b}{\partial u_2^2 \partial u_1} \\
& + 2646u_1^8 u_2 \frac{\partial^3 b}{\partial u_2^2 \partial u} + 2466u_1^7 u_3 \frac{\partial^2 b}{\partial u_2^2} + 378u_1^8 u_4 \frac{\partial^3 b}{\partial u_2^2} \quad (19)
\end{aligned}$$

It can be shown that the remaining compatibility conditions for the mixed derivatives of  $c$  follow from (10),(12),(15) and (19). Solving the system of partial differential equations (10),(12),(15) and (19) for the unknown function  $b(u, u_1, u_2, u_3, u_4)$  we arrive at the formula

$$b = \frac{1}{2u_1^6} (10u_1u_3 - 55u_2^2 + 2\alpha(u)u_1^4),$$

where  $\alpha(u)$  is an arbitrary function. Substituting the above expression for  $b$  into the conditions (9),(11),(13),(14),(16) and (17) and solving the resulting system of partial differential equations for the unknown function  $c(u, u_1, u_2, u_3, u_4, u_5, u_6)$ , we get

$$\begin{aligned}
c = & \frac{1}{u_1^8} \left( 3u_1^6 u_2 \frac{\partial \alpha(u)}{\partial u} + 2u_1^5 u_3 \alpha(u) - 6u_1^4 u_2^2 \alpha(u) + \beta(u)u_1^8 - 3u_1^3 u_5 + 65u_1^2 u_2 u_4 + 50u_1^2 u_3^2 \right. \\
& \left. - 615u_1 u_2^2 u_3 + 735u_2^4 \right),
\end{aligned}$$

where  $\beta(u)$  is another arbitrary function. □

Now let us turn to the property of having a momentum. Notice that the Fréchet derivative of the variational derivative of an arbitrary functional is a self-adjoint differential operator, see e.g. [13]. The following proposition states that every differential function  $h$  whose Fréchet derivative is a self-adjoint operator and which satisfies the condition  $\mathfrak{D}(h) = u_1$ , where  $\mathfrak{D}$  is a Hamiltonian operator with the leading coefficient of differential order less than or equal to 1, is of the form  $h = h(x, u)$ . It can be easily verified that any differential function of this form is the variational derivative of the functional  $\mathcal{P} = \int \int h(x, u) du dx$ . Thus, instead of looking for a functional  $\mathcal{P}$  such that  $\mathfrak{D}\delta_u \mathcal{P} = u_1$  we can study the existence of a differential function  $h(x, u)$  that satisfies the condition  $\mathfrak{D}(h) = u_1$ .

**Proposition 4.** *Let  $\mathfrak{D}$  be a fifth-order Hamiltonian operator whose leading coefficient is of differential order less than or equal to 1,*

$$\mathfrak{D} = aD_x^5 + D_x^5 \circ a + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c, \quad \text{ord}(a) \leq 1.$$



If there is a differential function  $h[u]$  such that  $\mathfrak{D}(h) = u_x$  and  $D_h = (D_h)^*$ , then  $h = h(x, u)$ .

*Proof.* As it was already mentioned above, the highest possible value  $m$  of the level of a fifth-order Hamiltonian operator is  $m = 7$ , so we have  $\text{ord}(a) \leq 1$ ,  $\text{ord}(b) \leq 4$  and  $\text{ord}(c) \leq 6$ .

Using relations for the coefficients  $a, b, c$  and their derivatives from [2] we see that  $\frac{\partial c}{\partial u_6} = 0$  and  $\frac{\partial b}{\partial u_4} = 0$ , so  $\text{ord}(c) \leq 5$  and  $\text{ord}(b) \leq 3$ . Suppose that  $\text{ord}(h) = K \geq 2$ . Then  $\text{ord}(D_x^5(h)) = 5 + K$  and differentiating the identity  $\mathfrak{D}(h) = u_x$  with respect to  $u_{5+K}$  we obtain

$$0 = \frac{\partial}{\partial u_{5+K}}(D_x^5(h)) = \frac{\partial h}{\partial u_K}.$$

Therefore,  $\text{ord}(h) \leq 1$ . Using the condition  $D_h = (D_h)^*$  we get  $\frac{\partial h}{\partial u_1} = 0$ , which was to be proved.  $\square$

**Proposition 5.** *No fifth-order Hamiltonian operator with the leading coefficient  $\pm 1$  has momentum.*

*Proof.* We have already noticed that it is sufficient to show that no fifth-order Hamiltonian operator with the leading coefficient 1 has a momentum because the operator with the leading coefficient  $-1$  can be transformed to this case by a special contact transformation which preserves the property of having (or not having) a momentum. So consider an operator with the leading coefficient 1 and suppose that this operator has a momentum. Then, differentiating the condition  $\mathfrak{D}(h) = u_1$  with respect to  $u_5$ , we get

$$\frac{\partial h}{\partial u} = -\frac{h}{2(u + \alpha)},$$

which implies that  $h(x, u) = f(x)/\sqrt{u + \alpha(x)}$ . Differentiating the condition  $\mathfrak{D}(h) = u_1$  with respect to  $u_3$  we arrive at  $\frac{\partial^2 f}{\partial x^2} = -f\gamma$ . Differentiating  $\mathfrak{D}(h) = u_1$  with respect to  $u_1$  and substituting for  $h$  and  $\frac{\partial^2 f}{\partial x^2}$ , we arrive at  $0 = 1$ , which is a contradiction. Thus, no fifth-order Hamiltonian operator with the leading coefficient  $\pm 1$  has momentum.  $\square$

**Proposition 6.** *Any fifth-order Hamiltonian operator with the leading coefficient  $\pm 1/u_1^4$  has a momentum, and the corresponding functional  $\mathcal{P}$  is of the form  $\mathcal{P} = \int \int h(u) du dx$ , where  $h(u)$  is a solution of the equation*

$$\pm \frac{\partial^5 h}{\partial u^5} + 2\alpha(u) \frac{\partial^3 h}{\partial u^3} + 3 \frac{\partial \alpha(u)}{\partial u} \frac{\partial^2 h}{\partial u^2} + 3 \frac{\partial^2 \alpha(u)}{\partial u^2} \frac{\partial h}{\partial u} + 2\beta(u) \frac{\partial h}{\partial u} + \frac{\partial^3 \alpha(u)}{\partial u^3} h + \frac{\partial \beta(u)}{\partial u} h - 1 = 0,$$

where  $\alpha(u)$  and  $\beta(u)$  are as in Lemma 3.

*Proof.* We will prove our claim only for the case of the leading coefficient  $1/u_1^4$ . The proof for the case of the leading coefficient equal to  $-1/u_1^4$  can be construed in a very similar fashion.

Suppose we are given an operator with the leading coefficient  $1/u_1^4$  which has a momentum. Differentiating the condition  $\mathfrak{D}(h) = u_1$  with respect to  $u_5$  we get the condition  $\frac{\partial h}{\partial x} = 0$ . Therefore, substituting for  $\frac{\partial h}{\partial x}$  in the condition  $\mathfrak{D}(h) = u_1$  we arrive at the equation given above.  $\square$

Combining Propositions 5 and 6 with the fact that special contact transformations preserve existence of momentum, we arrive at our main result.

- Theorem 1.** (1) *No fifth-order Hamiltonian operator that can be transformed under a special contact transformation to an operator with the leading coefficient  $\pm 1$  has momentum.*
- (2) *Any fifth-order Hamiltonian operator that can be transformed under a special contact transformation to an operator with the leading coefficient  $\pm 1/u_1^4$  has a momentum.*

## 5. EXAMPLES

**Example 1.** The operator

$$\mathfrak{D} = \frac{1}{2u_1^4} D_x^5 + D_x^5 \circ \frac{1}{2u_1^4} + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6} (10u_3u_1 - 55u_2^2 + u_1^4), \\ c &= \frac{1}{u_1^8} (u_1^5u_3 - 3u_1^4u_2^2 - u_1^8 - 3u_1^3u_5 + 65u_1^2u_2u_4 + 50u_1^2u_3^2 - 615u_1u_2^2u_3 + 735u_2^4), \end{aligned}$$

is of the form from Lemma 3 ( $\alpha = (1/2)$ ,  $\beta = -1$ ). The function  $h(u) = (-1/2)u$  is a solution of the ordinary differential equation

$$\frac{\partial^5 h}{\partial u^5} + \frac{\partial^3 h}{\partial u^3} - 2\frac{\partial h}{\partial u} - 1 = 0.$$

The functional  $\mathcal{P} = -(1/4) \int u^2 dx$  satisfies the condition  $\mathfrak{D}\delta_u \mathcal{P} = u_1$ .

**Example 2.** The operator

$$\mathfrak{D} = \frac{1}{2u_1^4} D_x^5 + D_x^5 \circ \frac{1}{2u_1^4} + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6} (10u_3u_1 - 55u_2^2 + 2\sin(u)u_1^4), \\ c &= \frac{1}{u_1^8} (3u_1^6u_2 \cos(u) + 2u_1^5u_3 \sin(u) - 6u_1^4u_2^2 \sin(u) + (\sin(u) + u)u_1^8 - 3u_1^3u_5 \\ &\quad + 65u_1^2u_2u_4 + 50u_1^2u_3^2 - 615u_1u_2^2u_3 + 735u_2^4), \end{aligned}$$

is of the form from Lemma 3 ( $\alpha = \sin(u)$ ,  $\beta = \sin(u) + u$ ). The function  $h(u) = 1$  is a solution of the ordinary differential equation

$$\frac{\partial^5 h}{\partial u^5} + 2\sin(u)\frac{\partial^3 h}{\partial u^3} + 3\cos(u)\frac{\partial^2 h}{\partial u^2} - \sin(u)\frac{\partial h}{\partial u} + 2\frac{\partial h}{\partial u} + h - 1 = 0.$$

The functional  $\mathcal{P} = \int u \, dx$  satisfies the condition  $\mathfrak{D}\delta_u \mathcal{P} = u_1$ .

## ACKNOWLEDGEMENTS

The author thanks Dr. A. Sergyeyev for stimulating discussions. This research was supported by the Silesian university in Opava under the student grant SGS/18/2010, by the Ministry of Education, Youth and Sports of the Czech Republic under the grant MSM 4781305904, and by the fellowship from the Moravian–Silesian region.

## REFERENCES

- [1] A. M. Astashov, A. M. Vinogradov, *On the structure of Hamiltonian operator in field theory*, J. Geom. and Phys. 3 (1986), no. 2, 263–287.
- [2] D. B. Cooke, *Classification Results and the Darboux Theorem for Low-Order Hamiltonian Operators*, J. Math. Phys. 32 (1991), 109–119.
- [3] A. de Sole, V. G. Kac, M. Wakimoto, *On Classification of Poisson Vertex Algebras*, Transformation Groups (2010), to appear, arXiv:1004.5387.
- [4] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, River Edge, NJ, 2003.
- [5] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley and Sons, Chichester etc., 1993.
- [6] B. A. Dubrovin and S. P. Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory* (Russian), Uspekhi Mat. Nauk 44, no.6, (1989), 29–98. English transl.: Russ. Math. Surv. 44 (1989) 35–124.
- [7] B. A. Dubrovin and S. P. Novikov, *Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method* (Russian), Dokl. Akad. Nauk SSSR 270 (1983), no. 4, 781–785.
- [8] A. S. Fokas, I. M. Gel'fand, *Bi-Hamiltonian structures and integrability*, in: Important developments in soliton theory, Springer, Berlin, 1993, 259–282.
- [9] A. S. Fokas, P. J. Olver, P. Rosenau, *A plethora of integrable bi-Hamiltonian equations*, in: Algebraic aspects of integrable systems, Birkhäuser Boston, Boston, MA, 1997, 93–101.
- [10] I. Krasil'shchik, *Algebraic Theories of Brackets and Related (Co)Homologies*, Acta Appl. Math. 109 (2010), 137–150, arXiv:0812.4676.
- [11] O. I. Mokhov, *Hamiltonian differential operators and contact geometry* (Russian), Funkc. anal. i ego prilož. 21 (1987), no.3, 53–60. English transl.: Funct. Anal. Appl. 21 (1987), 217–223.
- [12] O. I. Mokhov, *Symplectic and Poisson Geometry on Loop Spaces of Smooth Manifolds and Integrable Equations*, Harwood Academic Publishers, Amsterdam, 2001.
- [13] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, N.Y., 1993.
- [14] P. J. Olver, *BiHamiltonian Systems*, in: Ordinary and Partial Differential Equations, B.D. Sleeman and R.J. Jarvis, eds., Longman, N.Y. 1987, 176–193.
- [15] P. J. Olver, *Darboux' Theorem for Hamiltonian Differential Operators*, J. Diff. Equ. 71 (1988), 10–33.
- [16] P. J. Olver, *Dirac's Theory of Constraints in Field Theory and the Canonical Form of Hamiltonian Differential Operators*, J. Math. Phys. 27 (1986), 2495–2501.
- [17] A. Sergyeyev, *A Simple Way of Making a Hamiltonian System into a Bi-Hamiltonian One*, Acta Appl. Math. 83 (2004), 183–197, arXiv:nlin/0310012.
- [18] A. Sergyeyev, *Why nonlocal recursion operators produce local symmetries: new results and applications*, J. Phys. A: Math. Gen. 38 (2005), 3397–3407, arXiv:nlin/0410049.

MATHEMATICAL INSTITUTE, SILESIAN UNIVERSITY IN OPAVA, NA RYBNÍČKU 1, 746 01 OPAVA, CZECH REPUBLIC  
*E-mail address:* Jirina.Vodova@math.slu.cz